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Zero modes in a system of Aharonov–Bohm solenoids on the Lobachevsky plane

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Abstract
We consider a spin 1/2 charged particle on the Lobachevsky plane subjected to a magnetic field corresponding to a discrete system of Aharonov–Bohm solenoids. Let H^+ and H^- be the two components of the Pauli operator for spin-up and -down, respectively. We show that neither H^+ nor H^- has a zero mode if the number of solenoids is finite. On the other hand, a construction is described of an infinite periodic system of solenoids for which either H^+ or H^- has zero modes depending on the value of the flux carried by the solenoids.

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1. Introduction

Zero modes (wavefunctions of a quantum-mechanical Hamiltonian at zero energy) have applications to a wide range of branches of physics. In particular, the following areas are mentioned in [5] with connection to zero modes: chiral symmetry breaking in (1 + 1)-space-time quantum electrodynamics, edge states along the boundary of a disk threaded by a magnetic flux, singular contributions to the Hall conductance from electrons hopping on a square lattice in the presence of a uniform magnetic field, superconductivity of a cosmic string, localization of a fractional charge at a domain wall in a charge-density wave, induction of a persistent mass current in narrow-gap semiconductors, surface (or edge) states in a superconductor with a special symmetry, edge states in nanographite ribbon junctions, itinerant-electron ferromagnetism in the repulsive Hubbard model, random matrix theory and Anderson localization (see an extensive bibliography in [5, 13]). In all the cases, the appearance of zero modes leads to quantum fluctuations with important physical consequences.

We dwell on zero modes in the Hofstadter problem related to the Hall conductance quantization in a two-dimensional electron system [14]. In this case zero modes cause a topological singularity of the magnetic bands, and as a result, the Hall coefficient changes its behaviour from electronlike to holelike as the Fermi level crosses the zero energy. It is important to know how this behaviour is altered by the variation of the system parameters. In particular, it is interesting to study the impact of the magnetic field inhomogeneity and of the system curvature on the appearance of zero modes. Additional Aharonov–Bohm fluxes are frequently used tools to create magnetic field inhomogeneities [22]. Their influence on appearance of zero modes in a plane system was analysed in detail in [13]. On the other hand, the simplest way to take into account a non-trivial curvature of such a system is achieved by considering systems of constant curvature. Since complete surfaces of constant curvature have distinct topological properties depending on the curvature sign and, therefore, they require distinct mathematical tools for investigating the zero modes of Pauli operators, we restrict ourselves to a complete surface of negative constant curvature, i.e. to the Lobachevsky plane.

It is worth noting that the integer quantum Hall effect on the Lobachevsky plane was investigated with the help of non-commutative geometry in [4, 6, 7]. Moreover, an interesting model of the fractional quantum Hall effect has been proposed recently in [16]. It is based on the idea that, due to the effect of the strong electron–electron interaction, a single two-dimensional electron 'sees' the surrounding geometry as curved, with the moving electrons being arranged in a lattice in the hyperbolic plane. As a result, the single-electron problem on the Lobachevsky plane may be used to simulate the multi-electron problem on the Euclidean plane. On the other hand, two-dimensional electron systems with nontrivial curvature are technologically realizable [17] and widely studied both theoretically and experimentally [21]. In particular, surfaces of negative curvature can be used for describing the electron motion in the hyperfullerenes [19].

Being motivated by the above-mentioned problems, we consider a spin $1/2$ charged particle on the Lobachevsky plane subjected to a time-independent magnetic field corresponding to a discrete system of singular flux tubes perpendicular to the plane. We are interested in the zero modes of the Pauli operator

$$P = \begin{pmatrix} H^+ & 0 \\ 0 & H^- \end{pmatrix},$$

where H^+ and H^- are the spin-up and spin-down components, respectively. Since both H^+ and H^- are positive operators, zero modes are automatically ground states of the quantum system.

The current paper extends some results known for the Euclidean plane to a non-flat space having a constant curvature equal to -1. These results are based on the Aharonov–Casher observation [1] that the Pauli operators for spin $1/2$ particles in a magnetic field are related to factorable Schrödinger operators. It is well known that even in the case of a uniform magnetic field, the spectrum of the magnetic Schrödinger operator H changes drastically when changing the curvature of the base plane from zero to a constant negative value [8]. In particular, if the strength of the magnetic field is weak enough (more precisely, if the magnetic flux through a triangle with zero angles is less than one quantum), then the spectrum of H is purely absolutely continuous in contrast to the zero curvature case in which the spectrum is pure point. We show that the Pauli operator with a finite number of Aharonov–Bohm fluxes exhibits a similar behaviour: it has no zero modes on the Lobachevsky plane, whereas in the Euclidean case, the zero modes may exist in a finite system of solenoids, as analysed in [3]. In this connection, it is interesting to note that the constant negative curvature exerts no effect on the Berry phase for the zero-range potential well moving in the uniform constant magnetic field [2]. Furthermore, it has been shown in [12] that zero modes occur if the solenoids are arranged in an infinite plane lattice, and some generalizations and additional details of this result can be found in [13, 18]. Our theorem 8 is an extension of such results to the case of the Lobachevsky plane.

As was already mentioned, the approach we use is based on the Aharonov–Casher ansatz. This makes it possible to employ the theory of analytic functions when constructing the zero modes. Let us now describe the problem in more detail and introduce the basic notation. Some additional details related to this method are contained, e.g. in [9, 10].

Let M be an oriented Riemannian two-dimensional manifold with a conformal metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{\lambda^2(z,\bar{z})},$$

where $\lambda^2(z,\bar{z}) > 0$ (the function $\lambda^2(z,\bar{z})$ is called the *Poincaré metric*). The corresponding area 2-form is

$$\mathrm{d}\sigma = \frac{\mathrm{d}x\wedge\mathrm{d}y}{\lambda^2(z,\bar{z})} = \frac{\mathrm{i}}{2}\frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{\lambda^2(z,\bar{z})}.$$

By definition, a magnetic field on M is an exact 2-form $b = B\,\mathrm{d}\sigma$, where the real-valued (generalized) function B is called the *strength* of the field b. Since b is exact, we have $b = da$ where the 1-form $a = a_x\,\mathrm{d}x + a_y\,\mathrm{d}y = a_z\,\mathrm{d}z + a_{\bar{z}}\,\mathrm{d}\bar{z}$ is a vector potential of b. We set

$$a_z = \tfrac{1}{2}(a_x - \mathrm{i}a_y), \qquad a_{\bar{z}} = \tfrac{1}{2}(a_x + \mathrm{i}a_y).$$

Hence,

$$\lambda^{-2}B = \partial_x a_y - \partial_y a_x = \frac{2}{\mathrm{i}}(\partial_z a_{\bar{z}} - \partial_{\bar{z}} a_z).$$

We shall suppose that

$$a_x, a_y \in L^1_{\mathrm{loc}}(M, d\sigma) \cap C^\infty(M\backslash\Omega),$$

for some discrete subset Ω of M. Moreover, we suppose that each point of Ω is a point of discontinuity of a_x or a_y. Under these hypotheses, Ω is determined by a in a unique way. In particular, a_x or a_y may be the imaginary and the real part of a meromorphic function, respectively.

Let us define the following operators in $L^2(M, \mathrm{d}\sigma)$ with the domain $C_0^\infty(M\backslash\Omega)$:

$$P_x = -\mathrm{i}\partial_x - a_x \equiv -\mathrm{i}\nabla_x, \qquad P_y = -\mathrm{i}\partial_y - a_y \equiv -\mathrm{i}\nabla_y,$$
$$\nabla_z = \tfrac{1}{2}(\nabla_x - \mathrm{i}\nabla_y) = \partial_z - \mathrm{i}a_z, \qquad \nabla_{\bar{z}} = \tfrac{1}{2}(\nabla_x + \mathrm{i}\nabla_y) = \partial_{\bar{z}} - \mathrm{i}a_{\bar{z}},$$
$$T_\pm = P_x \pm \mathrm{i}P_y = -\mathrm{i}\nabla_x \pm \nabla_y.$$

Let us consider the quadratic form

$$h^\pm_{\max}(f) = \int_M \lambda^2 |T_\pm f|^2\,\mathrm{d}\sigma$$

with the domain

$$Q\big(h^\pm_{\max}\big) = \left\{ f \in L^2(M,\mathrm{d}\sigma);\ \nabla_x f, \nabla_y f \in L^1_{\mathrm{loc}}(M\backslash\Omega, \mathrm{d}\sigma), \text{ and } \int_M \lambda^2 |T_\pm f|^2\,\mathrm{d}\sigma < \infty \right\}.$$

The quadratic form $h^\pm_{\max}$ is closed and defines a self-adjoint operator $H^\pm$ in $L^2(M, \mathrm{d}\sigma)$. On $C_0^\infty(M\backslash\Omega)$, we have

$$\lambda^2 T_+ T_- = H^-, \qquad \lambda^2 T_- T_+ = H^+,$$

and

$$\lambda^{-2} H^\pm = P_x^2 + P_y^2 \mp \lambda^{-2} B.$$

Clearly, both H^+ and H^- are positive operators.

Suppose that in the sense of distributions

$$\lambda^{-2}B = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} \equiv \Delta\varphi$$

where φ is a regular distribution (a locally integrable function). Then for the vector potential one can choose

$$a_{\bar{z}} = \mathrm{i}\partial_{\bar{z}}\varphi, \qquad a_z = -\mathrm{i}\partial_z\varphi,$$

and the zero modes of H^+ (resp. H^-), i.e. L^2-solutions $\psi \neq 0$ to the equation $H^\pm\psi = 0$, have the form

$$\psi(z, \bar{z}) = \exp(\mp\varphi(z, \bar{z}))f(z, \bar{z}),$$

where f is a holomorphic (resp. antiholomorphic) function on $M\backslash\Omega$.

2. Finite number of Aharonov–Bohm solenoids

In what follows M will be the Lobachevsky plane which we shall model as the disc

$$\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\} \qquad \text{with} \qquad \lambda = \frac{1 - z\bar{z}}{2}.$$

Equivalently, one could model M as the upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C}; \operatorname{Im} z > 0\}$ with $\lambda = (z - \bar{z})/(2\mathrm{i})$.

Proposition 1. *Let B be the magnetic field on M corresponding to a finite family of Aharonov–Bohm solenoids with non-zero fluxes. Then $H^\pm$ has no zero modes.*

Proof. Let us consider the operator H^+; the proof is similar in the case of H^-. Let $a_k \in \mathbb{D}$, $k = 1, \ldots, n$, be a finite set of points. Consider the function

$$\varphi(z, \bar{z}) = \prod_{k=1}^{n} |z - a_k|^{\theta_k}.$$

Then

$$\Delta \log(\varphi) = 2\pi \sum_{k=1}^{n} \theta_k \delta(z - a_k),$$

and the corresponding field strength equals

$$B(z, \bar{z}) = \frac{\pi}{2} \sum_{k=1}^{n} \theta_k (1 - |a_k|^2)^2 \delta(z - a_k).$$

Let us note that for the field $B = \frac{\pi}{2}\theta(1 - |a|^2)^2\delta(z - a)$, the flux equals

$$\Phi = \frac{1}{2\pi} \int_M B \, \mathrm{d}\sigma = \theta.$$

As usual, due to the gauge symmetry one can assume that $0 < \theta_k < 1$ for all k. Let us suppose that H^+ has a zero mode ψ. Then

$$\psi(z, \bar{z}) = \prod_{k=1}^{n} |z - a_k|^{-\theta_k} f(z), \tag{1}$$

where f is holomorphic on the domain $\mathbb{D}\backslash\{a_1, \ldots, a_n\}$. Since $\psi \in L^2(\mathbb{D}, d\sigma)$, the function f cannot have a pole nor an essential singularity at any of the points $a_1, \ldots, a_n$, and therefore, f has an analytic extension to the whole domain $\mathbb{D}$. Moreover, from (1) one deduces that $|f(z)| \leqslant \text{const}|\psi(z, \bar{z})|$ on $\mathbb{D}$ and therefore $f \in L^2(\mathbb{D}, d\sigma)$. Since this means that f^2 is a holomorphic function on $\mathbb{D}$ belonging to $L^1(\mathbb{D}, d\sigma)$, the following lemma completes the proof. □

Lemma 2. *Let f be a holomorphic function on $\mathbb{D}$. If $f \in L^1(\mathbb{D}, d\sigma)$ then $f = 0$.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and suppose that the series converges in $\mathbb{D}$. Denote $z = |z|\,\mathrm{e}^{\mathrm{i}\varphi}$. The functions $\mathrm{e}^{-\mathrm{i}n\varphi} f(z)$ belong to $L^1(\mathbb{D}, d\sigma)$ for all $n \in \mathbb{Z}$. Moreover, for $n \geqslant 0$ we have

$$\int_{\mathbb{D}} \mathrm{e}^{-\mathrm{i}n\varphi} f(z) d\sigma = \lim_{r\to 1-} \int_{|z|<r} \mathrm{e}^{-\mathrm{i}n\varphi} f(z) d\sigma = 8\pi a_n \lim_{r\to 1-} \int_0^r \frac{\rho^{n+1}}{(1-\rho^2)^2} d\rho.$$

Since the last integral diverges as $r \to 1-$, it necessarily holds $a_n = 0$. □

Remark 3. On the Euclidean plane $\mathbb{R}^2$, the following Aharonov–Casher theorem is valid [1]: *if $B(x, y)$ is a 'regular' function with a compact support then* $\dim \mathrm{Ker}(H^+ \oplus H^-) = \langle|\Phi|\rangle$, *where*

$$\Phi = \frac{1}{2\pi} \int_{\mathbb{R}^2} B \,\mathrm{d}x\,\mathrm{d}y$$

is the magnetic flux, and for $x \geqslant 0$,

$$\langle x \rangle = \begin{cases} [x], & \text{if } x \notin \mathbb{Z}, \\ x-1, & \text{if } x \in \mathbb{Z} \quad \text{and} \quad x > 0, \\ 0, & \text{if } x = 0, \end{cases}$$

(here $[x]$ stands for the integer part of x). The following example shows that an analogous statement is not true for the Lobachevsky plane.

Let $M = \mathbb{D}$ and $B(z, \bar{z}) = \lambda^2(|z|)F(|z|)$, where

$$F(r) = \begin{cases} \widetilde{B}, & \text{if } 0 \leqslant r \leqslant r_0, \\ 0, & \text{if } r_0 < r < 1, \end{cases}$$

(here $\widetilde{B}$ is a positive number and r_0, $0 < r_0 < 1$, is fixed). To find a function φ such that $\Delta\varphi = F$, one has to solve the equation

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} r \frac{\mathrm{d}}{\mathrm{d}r}\varphi(r) = F(r).$$

It is easy to show that we can set

$$\varphi(r) = \begin{cases} \dfrac{\widetilde{B}}{4} r^2, & \text{if } 0 \leqslant r \leqslant r_0, \\ \dfrac{\widetilde{B}}{4} r_0^2 + \dfrac{\widetilde{B}}{2} r_0^2 \log\left(\dfrac{r}{r_0}\right), & \text{if } r_0 < r < 1. \end{cases}$$

It is clear that for every $\widetilde{B} > 0$, we have

$$\inf_{0 \leqslant r \leqslant 1} \exp(\mp\varphi(r)) > 0.$$

This implies that if $f \exp(\mp\varphi)$ is square integrable then the same is true for f. By lemma 2, for every function $f \neq 0$ which is holomorphic (antiholomorphic) on $\mathbb{D}$ it holds $f \exp(\mp\varphi) \notin L^2(\mathbb{D}, d\sigma)$. Hence $\dim \mathrm{Ker}(H^+ \oplus H^-) = 0$. On the other hand, the flux

$$\Phi = \frac{1}{2\pi} \int_{\mathbb{D}} B \,\mathrm{d}\sigma = \frac{1}{2\pi} \int_{\mathbb{D}} F(r) \,\mathrm{d}x\,\mathrm{d}y = \frac{\widetilde{B}}{2} r_0^2$$

can be an arbitrary positive number. Note that for the Dirac operator on the Lobachevsky plane, the similar result is obtained in [20].

3. An infinite system of Aharonov–Bohm solenoids

Here we consider magnetic fields with infinite total fluxes, more precisely, we consider an infinite set of Aharonov–Bohm solenoids penetrating the Lobachevsky plane in the points of a discrete set Γ in such a way that the resulting field is invariant with respect to a discrete group G of isometries of $\mathbb{D}$. In order to approach the problem more easily, we suppose that the action of G on $\mathbb{D}$ is co-compact which means that the factor space $\mathbb{D}/G$ is compact. The construction of the system of fluxes is as follows. We choose a fundamental domain $F \subset \mathbb{D}$ with respect to the action of G, we fix a finite subset $\mathrm{K} \subset \mathrm{F}$, and we set $\Gamma = \bigcup_{g\in G} g\mathrm{K}$. For each $\kappa \in \mathrm{K}$, we choose a non-integer flux θ_κ. The resulting magnetic field consists of all Aharonov–Bohm solenoids penetrating the plane at the points $\gamma \in \Gamma$. If $\gamma = g\kappa$, for some $g \in G$ and $\kappa \in K$, then the flux corresponding to γ equals θ_κ.

As an illustration of the Aharonov–Casher method, we start from a short remark where we discuss a known result concerning a uniform magnetic field on the Lobachevsky plane M.

Remark 4. Suppose that $B = \mathrm{const}$ and without loss of generality we can assume that $B > 0$. It is known (see [8]) that in this case the spectrum of $H^\pm$ is purely absolutely continuous if and only if $B \leqslant 1/2$. If it is the case then the spectrum consists of the semi-axis $[1/4 + B^2 \mp B, +\infty[$. Otherwise, in addition to the semi-axis, the spectrum of $H^\pm$ contains infinitely degenerate eigenvalues $E_n = B(2n + 1 \mp 1) - n^2 - n$, where $n \in \mathbb{Z}$ and $0 \leqslant n < B - 1/2$. From here one deduces that the operator H^+ has zero modes if and only if $B > 1/2$ while $H^- \geqslant 2B$ never has zero modes. As a demonstration of the effectiveness of the Aharonov–Casher method let us re-establish the observation concerning zero modes of H^+.

First, we find a function φ defined in $\mathbb{D}$ such that

$$\Delta\varphi = B\lambda^{-2}.$$

Assuming that φ depends on $|z|$ only, we arrive at the equation

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{\mathrm{d}r}\varphi(r) = B\lambda(r)^{-2}.$$

Its solution reads

$$\varphi(r) = -B\log(1 - r^2).$$

The operator H^+ has a zero mode if and only if there exists a function $f \neq 0$ which is holomorphic on $\mathbb{D}$ and such that

$$(1 - r^2)^{2B}\frac{1}{(1 - r^2)^2}|f(z)|^2 \in L^1(\mathbb{D}, \mathrm{d}x \wedge \mathrm{d}y). \tag{2}$$

It is clear that in the case when $B > 1/2$ all functions f which are holomorphic on $\mathbb{D}$ and bounded on $\overline{\mathbb{D}}$ satisfy condition (2). On the other hand, suppose that a function $f(z)$ is holomorphic on $\mathbb{D}$ and satisfies condition (2). Denote $g(z) = f(z)^2 = \sum_{m=0}^{\infty} a_m z^m$. Then for every $n \in \mathbb{Z}, n \geqslant 0$,

$$\begin{aligned}\int_0^{2\pi}\int_0^1 &(1 - r^2)^{2B}\frac{1}{(1 - r^2)^2}g(z)\,\mathrm{e}^{-\mathrm{i}n\varphi}r\,\mathrm{d}r \wedge \mathrm{d}\varphi \\ &= 2\pi a_n \lim_{\rho\to 1-}\int_0^\rho (1 - r^2)^{2B}\frac{r^{n+1}}{(1 - r^2)^2}\,\mathrm{d}r.\end{aligned} \tag{3}$$

By assumption, the integral on the lhs in (3) is finite while the integral on the rhs converges as $\rho \to 1-$ if and only if $B > 1/2$. Hence $f(z)$ necessarily vanishes everywhere on $\mathbb{D}$ if $B \leqslant 1/2$.

Let us now return to the construction of a symmetric lattice of Aharonov–Bohm fluxes. We shall need the following lemma.

Lemma 5. *Let G be a discrete co-compact group of isometries acting on the disc $\mathbb{D}$ equipped with the Poincaré metric $\mathrm{d}s^2$, and let F be a precompact fundamental domain for G. Choose an element z_γ in each domain γF, $\gamma \in G$. If $d \geqslant 2$, then*

$$\sum_{\gamma\in G}(1-|z_\gamma|^2)^d < \infty. \tag{4}$$

Proof. It is sufficient to prove the lemma for $d = 2$. Let us fix ε, $0 < \varepsilon < 1/2$. Consider a finite family $\{S_j\}_{j=1}^m$ of nonempty measurable mutually disjoint subsets $S_j \subset F$ such that

(1) $\bigcup_{j=1}^m S_j = F$,
(2) $\operatorname{diam} S_j \leqslant \varepsilon, \forall j$,
(3) $\sigma(S_j) = \frac{1}{m}\sigma(F)$,

(σ stands for the area). Denote by $m_{j\gamma}$ (resp. $M_{j\gamma}$) the infimum (resp. the supremum) of the function $h(z,\bar{z}) = (1-|z|^2)^2$ on the set $\overline{\gamma S_j}$. It is sufficient to verify that

$$\sum_{j=1}^m \sum_{\gamma\in G} M_{j\gamma} < \infty.$$

It is convenient to employ the polar geodesic coordinates (ρ,θ) on $\mathbb{D}$ centred at $z = 0$. If $z = r\,\mathrm{e}^{\mathrm{i}\varphi}$, then

$$r = \operatorname{th}\left(\frac{\rho}{2}\right), \qquad \varphi = \theta.$$

In these coordinates,

$$h(\rho,\theta) = \operatorname{ch}\left(\frac{\rho}{2}\right)^{-4}.$$

From the triangle inequality, it follows that for any couple of points from $\mathbb{D}$ it holds

$$|\rho_1-\rho_2| \leqslant \operatorname{dist}((\rho_1,\theta_1),(\rho_2,\theta_2))$$

(where $\operatorname{dist}(\cdot,\cdot)$ is the distance in the Lobachevsky plane) and therefore

$$\sup\{|\rho_1-\rho_2|;(\rho_1,\theta_1),(\rho_2,\theta_2)\in\gamma S_j\} \leqslant \varepsilon.$$

Since h is independent of θ, we have

$$\begin{aligned} M_{j\gamma} - m_{j\gamma} &\leqslant \varepsilon \sup\left\{\left|\frac{\mathrm{d}}{\mathrm{d}\rho}\operatorname{ch}\left(\frac{\rho}{2}\right)^{-4}\right|;(\rho,\theta)\in\gamma S_j\right\} \\ &= 2\varepsilon\sup\left\{\operatorname{ch}\left(\frac{\rho}{2}\right)^{-4}\operatorname{th}\left(\frac{\rho}{2}\right);(\rho,\theta)\in\gamma S_j\right\} \\ &\leqslant 2\varepsilon\sup\left\{\operatorname{ch}\left(\frac{\rho}{2}\right)^{-4};(\rho,\theta)\in\gamma S_j\right\} \\ &= 2\varepsilon M_{j\gamma}. \end{aligned}$$

Consequently,

$$M_{j\gamma} \leqslant \frac{m_{j\gamma}}{1-2\varepsilon} \leqslant \frac{m}{(1-2\varepsilon)\sigma(F)}\int_{\gamma S_j} h(\rho,\theta)\,\mathrm{d}\sigma$$

and so

$$\sum_{j\gamma} M_{j\gamma} \leqslant \frac{m}{(1-2\varepsilon)\sigma(F)} \int_{\mathbb{D}} h(\rho,\theta)\,\mathrm{d}\sigma = \frac{4m\pi}{(1-2\varepsilon)\sigma(F)}. \tag{5}$$

This proves the lemma. □

Remark 6. If the points z_γ are congruent modulo G then inequality (4) is well known and it is true for every discrete group G (see [15, lemma III.5.2]).

Remark 7. Let $K = -1$ be the Gaussian curvature of the Lobachevsky plane and let g be the genus of the closed surface $\mathbb{D}/G$. The Gauss–Bonnet formula tells us that

$$\frac{1}{2\pi}\int_{\mathbb{D}/G} K\,\mathrm{d}\sigma = -\frac{1}{2\pi}\sigma(F) = 2-2g.$$

Hence $g \geqslant 2$ and we have $\sigma(F) \geqslant 4\pi$ independently of the group G. Moreover, we can choose

$$m = \left[\frac{\sigma(F)}{\varepsilon}\right] + 1.$$

With this choice, the rhs of (5) can be further estimated from above by the expression

$$\frac{1}{1-2\varepsilon}\left(\frac{4\pi}{\varepsilon}+1\right)$$

which is already independent of G. In particular, for $\varepsilon = 1/4$, we get the upper bound $32\pi + 2$. In the case of arbitrary $d \geqslant 2$, we have the estimate

$$\sum_{\gamma\in G}(1-|z_\gamma|^2)^d < \frac{4m\pi}{(1-d\varepsilon)(d-1)\sigma(F)},$$

where $\varepsilon < 1/d$ and the rhs can be again replaced by an expression independent of G.

Recall that the group of motions of $\mathbb{D}$ regarded as the Lobachevsky plane is $SU(1,1)$, the group of transformations

$$Az = \frac{az+b}{\bar{b}z+\bar{a}}, \qquad \text{where} \quad |a|^2 - |b|^2 = 1.$$

Let G be a discrete co-compact subgroup of $SU(1,1)$ and let F be a precompact fundamental domain of G. Suppose that $W(z)$ is an automorphic form on $\mathbb{D}$ of weight $2k$, $k \geqslant 1$, with respect to G, i.e. $W(z)$ is a meromorphic function on $\mathbb{D}$ obeying the following condition:

$$\forall A \in G, \qquad W(Az) = A'(z)^{-k}W(z). \tag{6}$$

For simplicity, we restrict ourselves to the case when W has only simple poles and zeros. Let us note that if G is a discrete group then automorphic forms do indeed exist, see for example [15, chapter III].

We can choose F in such a way that ∂F contains no poles nor zeros of W. Let $a_1, \ldots, a_n$ be the set of all zeros and let $b_1, \ldots, b_m$ be the set of all poles of W in F. It is known that $n > m$ (see [11, section 49, theorem 4]). Then the function $B = \theta\lambda^{-2}\Delta\log(|W|)$, $\theta \in \mathbb{R}$, is the strength of the magnetic field of a system of Aharonov–Bohm solenoids intersecting the Lobachevsky plane at the points γa_j and γb_j, where γ is an arbitrary transformation from G. A solenoid intersecting the plane at γa_j carries the flux θ, and a solenoid intersecting the plane at γb_j carries the flux $-\theta$.

Using the gauge symmetry we again assume, without loss of generality, that $0 < \theta < 1$.

Theorem 8. *If* $k\theta \geqslant 1$, *then the operator* $H^+(B)$ *has zero modes. If* $0 < k\theta < k - 1$, *then the operator* $H^-(B)$ *has zero modes.*

Proof. We restrict ourselves to the case of operator H^+; the proof is similar for H^-. To prove the claim one has to find a function $f(z)$ analytic in $\mathbb{D}$ such that the function

$$\psi(z,\bar{z}) = f(z)|W(z)|^{-\theta}$$

belongs to $L^2(\mathbb{D}, \mathrm{d}\sigma)$.

One can easily check that

$$\forall A \in SU(1,1), \qquad \lambda(Az, \overline{Az}) = |A'(z)|\lambda(z,\bar{z}). \tag{7}$$

From (6) and (7), it follows that:

$$|W(z)| = (1 - |z|^2)^{-k} r(z,\bar{z})$$

where $r(z,\bar{z})$ is a G-periodic function. Hence

$$|W(z)|^{-2\theta} = (1 - |z|^2)^{2k\theta} r(z,\bar{z})^{-2\theta}.$$

It is clear that $r^{-2\theta} \in L^1(F, \mathrm{d}\sigma)$ ($W(z)$ has only simple zeros and so the singularities of $r(z,\bar{z})^{-2\theta}$ are integrable). Consequently, for every function f which is bounded and analytic on $\mathbb{D}$, we have

$$\begin{aligned}\int_{\mathbb{D}} |f(z)|^2 |W(z)|^{-2\theta}\, \mathrm{d}\sigma &= \sum_{\gamma\in G} \int_{\gamma F} |f(z)|^2 (1 - |z|^2)^{2k\theta} r(z,\bar{z})^{-2\theta}\, \mathrm{d}\sigma \\ &\leqslant \|f\|_\infty \int_F r(z,\bar{z})^{-2\theta}\, \mathrm{d}\sigma \sum_{\gamma\in G} (1 - |z_\gamma|^2)^{2k\theta}\end{aligned}$$

where z_γ is a point from $\overline{\gamma F}$. By lemma 5 $\sum_{\gamma\in G}(1 - |z_\gamma|^2)^{2k\theta} < \infty$. This completes the proof. □

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